

# Erdős's Matching Conjecture and $s$ -wise $t$ -intersection Conjecture via Symmetrical Smoothing Method

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## Abstract

We find the formula for the maximal cardinality of the family of  $n$ -tuples from  $\binom{[n]}{k}$  which does not have  $\ell$ -matching. This formula after some analytical issues can be reduced to the Erdős's Matching formula. Also we prove the conjecture about the cardinality of maximal  $s$ -wise  $t$ -intersecting family of  $k$ -element subsets of  $[n]$ . In the proofs we use original method which we have already used in the proof of Miklós-Manikam-Singhi conjecture in [1]. We call this method Symmetrical smoothing method.

## I Introduction and Formulation of Results

Define  $[n] = \{1, \dots, n\}$  and  $\binom{[n]}{k} = \{E \subset [n] : |E| = k\}$ . We say that family  $\mathcal{A} \subset \binom{[n]}{k}$  has  $\ell$ -matching if there exists the set  $\{E_i, i \in [\ell]\} \subset \mathcal{A}$  such that  $E_i \cap E_j = \emptyset$  when  $i \neq j$ .

First problem which we would like to introduce is to find the maximal cardinality  $M(\ell, n, k)$  of  $\mathcal{A} \subset \binom{[n]}{k}$  which has no  $\ell$ -matching.

In 1965 Erdős [2] formulated the following

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**Conjecture 1** *The value  $M(\ell, n, k)$  satisfies the following equality*

$$M(\ell, n, k) = \max \left\{ \binom{k\ell - 1}{k}, \binom{n}{k} - \binom{n - \ell + 1}{k} \right\}. \quad (1)$$

This conjecture is one of the main statements in extremal hypergraph theory. Erdős wrote in [2] that he manage to prove this corollary for  $k = 2$  and for  $\ell = 2$  it is Erdős-Ko-Rado result but general case seems elusive.

Later this corollary was confirmed for several conditions on parameters of the problem, we mention the proof of the conjecture for  $n \geq (2\ell - 1)k - \ell + 1$  in [4], there was proved that in this case

$$M(\ell, n, k) = \binom{n}{k} - \binom{n - \ell + 1}{k}.$$

Also the conjecture was proved for  $k = 3$  in [7].

Let's mention also that asymptotic equality for  $M(\ell, n, k)$ , which follows from the conjecture is proved for some parameters in [8].

First our result is the proof of the following

**Lemma 1** *The following equality is valid*

$$M(\ell, n, k) = \max_{1 \leq i \leq k} \sum_{j \geq i} \binom{\ell i - 1}{j} \binom{n - \ell i + 1}{k - j}. \quad (2)$$

Thus the proof that the Conjecture 1 is true for all parameters  $\ell, n, k$  reduced to the proof of technical equality

$$\begin{aligned} & \max_{1 \leq i \leq k} \sum_{j \geq i} \binom{\ell i - 1}{j} \binom{n - \ell i + 1}{k - j} \\ &= \max \left\{ \binom{k\ell - 1}{k}, \binom{n}{k} - \binom{n - \ell + 1}{k} \right\}. \end{aligned}$$

Note, that for the arbitrary  $i \in [k]$  the choice of the set

$$\mathcal{A} = \left\{ A \in \binom{[n]}{k} : |A \cap [\ell i - 1]| \geq i \right\}$$

shows that

$$M(\ell, n, k) \geq \max_{1 \leq i \leq k} \sum_{j \geq i} \binom{\ell i - 1}{j} \binom{n - \ell i + 1}{k - j}.$$

So we need to prove the opposite inequality

To introduce our second result we introduce some additional notations. We say that family  $\mathcal{B} \subset \binom{[n]}{k}$  is  $s$ -wise  $t$ -intersecting if for the arbitrary subset  $\{E_i, i \in [s]\} \subset \mathcal{B}$  the following relation is true  $|E_1 \cap E_2 \cap \dots \cap E_s| \geq t$ . Let  $N(s, n, k, t)$  is the maximal cardinality of  $s$ -wise  $t$ -intersecting family from  $\binom{[n]}{k}$ .

There is the following old

**Conjecture 2** *Let  $sk < (s-1)n + t$ . The following equality is valid*

$$N(s, n, k, t) = \max_{r \geq 0} \left\{ \left| E \in \binom{[n]}{k} : |E \cap [t + rs]| \geq t + (s-1)r \right| \right\}.$$

Note that choice of the set

$$\left\{ \left| E \in \binom{[n]}{k} : |E \cap [t + rs]| \geq t + (s-1)r \right| \right\}$$

for  $r \geq 0$  shows that

$$N(s, n, k, t) \geq \max_{r \geq 0} \left\{ \left| E \in \binom{[n]}{k} : |E \cap [t + rs]| \geq t + (s-1)r \right| \right\}.$$

So we need to prove the opposite inequality.

Note also that if  $sk \geq (s-1)n + t$ , then the whole set  $\binom{[n]}{k}$  is  $s$ -wise  $t$ -intersecting family. There are many publications which are devoted to solution of this problem in particular cases. The most important result was obtained by Ahlswede and Khachatrian in celebrated paper [9]. They confirm the validness of this conjecture for the case  $s = 2$ . In all other cases there are partial solutions (when some parameters are given and  $n$  is sufficiently large). We mention papers [10]-[11].

Our second result is the proof of this conjecture for all parameters  $s, n, k$ .

We note that in [12] we prove the fractional analog of lemma 1. Hence we have confirmed the expression similar to (2) for fractional matching also.

The paper is organized as follows: in Section II we introduce the *Symmetrical smoothing method*, which actually we have already used in [1] and in [12]. We also formulate and prove technical lemma 2 which we use also later, in section III. In Section III we, using lemma 2, complete the proof of conjecture 2. In the proof of lemma 1 in section II we use lemma 2.

## Section II

Next we use the natural bijection between  $2^{[n]}$  and set of binary  $n$ -tuples  $\{0, 1\}^n$  and make no difference between these two sets.

We say that family  $\mathcal{A} \subset \binom{[n]}{k}$  is (left) compressed if from the inclusion  $A = (a_1, \dots, a_k) \in \mathcal{A}$  and the conditions  $b_j \leq a_j$  follows that  $B = (b_1, \dots, b_k) \in \mathcal{A}$ . Note, that we can assume that the extremal intersection families are left compressed.

Also we can assume that left compressed family  $\mathcal{A} \subset \binom{[n]}{k}$  is defined by the inequalities

$$\mathcal{A} = \left\{ x \in \binom{[n]}{k} : (\omega_i, x) > 0, i \in [N] \right\}, \quad (3)$$

where

$$\omega_i = (\omega_{i,1}, \dots, \omega_{i,n}) \in R^n$$

and

$$\omega_{i,j} \geq \omega_{i,j+1} \quad (4)$$

when  $j \in [n-1]$ . Indeed, the set  $\mathcal{A}$ , which defined by the inequalities (3) is shifted. Arbitrary left compressed set can be defined as the intersection of the sets, which determined by the inequalities (3) (with different omegas). However we will see later that we can restrict ourselves assuming that set is generated by only one inequality.

Next we assume that extremal families in both problems are left compressed and are defined by one inequality from (3) and condition (4) is satisfied. It is easy to see that if the family  $\mathcal{A}$  has  $\ell$ -matching then the non intersecting set  $(x_1, \dots, x_\ell) \subset \mathcal{A}$  can be chosen in such way that  $x_i \subset [\ell k]$ .

The Symmetrical Smoothing Method consists in approximation of the number  $|\mathcal{A}|$  by the smooth symmetric function of  $\omega$ , which allows to use analytic methods to determine the values of  $\omega_j$  on which achieves extremum of  $|\mathcal{A}|$ .

Some of the values  $\omega_j$  can be negative. Next we make transformations of  $\omega$  and write the system (3) in the equivalent form, where coefficients are all nonnegative. Consider the following set of basis (for the representation of  $\omega$ ) vectors  $z_j = (k\ell - dj, \dots, k\ell - dj, -dj, \dots, -dj) \in R^n$ , where the number of coordinates  $k\ell - dj$  is equal to  $j$  and  $j \in [k\ell - 1]$ . Because the maximal set  $\mathcal{A}$  is compressed, we need to choose only first  $k\ell$  coordinates  $\omega_{i,j}$  and other we can choose as large as possible, i.e. all of them are equal to  $\omega_{k\ell}$ .

Then it is easy to check, that vectors  $\omega$ , which coordinates satisfy inequalities (4) and which determine the maximal family in first problem can be represent as the sum

$$\omega = \sum_{j=1}^{k\ell-1} \alpha_j z_j \quad (5)$$

with non negative coordinates  $\alpha_j \geq 0$  and some  $d$ . Indeed from (5) follows that for  $j \leq k\ell - 1$

$$\omega_j - \omega_{j+1} = \alpha_j k\ell$$

or

$$\alpha_j = \frac{\omega_j - \omega_{j+1}}{k\ell} \geq 0.$$

The last equation contains only differences of  $\omega_j$  we have one degree of freedom to determine  $\omega_{k\ell}$ , to do this we choose proper  $d_i$ . It can be easily shown, that

$$d = \frac{k\ell}{n} - \frac{\sum_{j=1}^n \omega_j}{n \sum_{j=1}^{k\ell-1} j \alpha_j}.$$

Substituting expansion (5) to the inequality  $(\omega, x) \geq 0$  we obtain

$$\begin{aligned} \left( \sum_{j=1}^{k\ell-1} \alpha_j z_j, x \right) &= \sum_{j=1}^{k\ell-1} \alpha_{i,j} \left( k\ell \sum_{m=1}^j x_m - jkd \right) \\ &= k\ell \sum_{j=1}^{k\ell-1} \left( \sum_{m=j}^{k\ell-1} \alpha_m \right) x_j - kd \sum_{j=1}^{k\ell-1} \alpha_j j \geq 0. \end{aligned} \quad (6)$$

Define

$$\beta_j = \frac{\sum_{m=j}^{k\ell-1} \alpha_m}{\sum_{m=1}^{k\ell-1} m \alpha_m}.$$

We can rewrite inequality in (6) as follows

$$\sum_{j=1}^{k\ell-1} \beta_j x_j \geq \delta, \quad (7)$$

where  $\beta_j \geq 0$  and  $\beta_j \geq \beta_{j+1}$ . Without loss of generality we can assume that  $\delta > 0$ , otherwise inequality (7) does not impose any restriction on the choice of  $x$ .

Thus the maximal family  $\mathcal{A}$  which does not has  $\ell$ -matching can be determined by the system of the inequalities

$$\sum_{j=1}^{k\ell-1} \beta_j x_j \geq \delta, \quad (8)$$

for  $i \in [N]$  and  $\delta > 0$  and the choice of  $\beta_j$  for  $j \in [k\ell - 1]$  is such that

$$\beta_j \geq \beta_{j+1}, \beta_j \geq 0, \sum_{j=1}^{k\ell-1} \beta_j = 1. \quad (9)$$

Also note that we can assume that for the  $\beta_i$  which determine the maximal family  $\mathcal{A}$

$$\left| \sum_{j=1}^{k\ell-1} \beta_j x_i - \delta \right| > \delta$$

for all  $i$  and  $x \in \binom{[n]}{k}$  and sufficiently small  $\delta > 0$ . This is because the number of  $k$ -element subset of  $[n]$  is finite set so as the number of relations (8) which determine  $\mathcal{A}$  and we can vary coordinates of  $\beta_i$  in small range without changing the family which is determined by equations (8). Because of this we can also assume that there are strict inequality in (8).

Define

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{\xi^2}{2}} d\xi.$$

It is easy to see that  $x \in \mathcal{A}$  if and only if for the arbitrary small but fixed  $\mu > 0$ , and sufficiently small  $\sigma = \sigma(\mu) > 0$  the following inequality is satisfied

$$Z(x, \sigma) = \varphi((\beta, x) - \delta)/\sigma) > 1 - \mu. \quad (10)$$

Define

$$Y(\mathcal{A}) = \sum_{x \in \binom{[n]}{k}} Z(x, \sigma).$$

We can approximate  $|\mathcal{A}|$  as follows:

$$||\mathcal{A}| - Y(\mathcal{A})| < \epsilon_1, \quad (11)$$

where  $\epsilon_1 > 0$  can be chosen arbitrary small. Hence to find the maximum of  $|\mathcal{A}|$  is equivalent to find the maximum of  $Y(\mathcal{A})$  over the choice allowed for  $\beta$  and  $\delta$ .

Next we show that maximum of  $Y(\mathcal{A})$  achieved on step functions  $\beta$  i.e. when  $\beta_j = \frac{1}{a}$  for  $j \in [a]$  for some  $a \in [k\ell - 1]$ .

**Lemma 2** *If we impose condition that  $\beta_j = 0$  when  $j = a + 1, \dots, k\ell - 1$ , then varying  $\delta$  in arbitrary small range we can achieve the situation that maximum of  $Y(\mathcal{A})$  achieved when  $\beta_j = \frac{1}{a}$  for  $j \in [a]$ .*

**Proof.** Next we will consider the case when  $a_i > 4$  for all  $i$ . The case  $a_i \leq 4$  is easy. Proof is made by differentiation over  $\beta_j$  with conditions  $\beta_j = 0$  for  $j = a + 1, \dots, k\ell - 1$ . Let's make this procedure. Because  $\beta_a = 1 - \sum_{j=1}^{a-1} \beta_j$  we have for  $j \in [a - 1]$

$$Y'_{\beta_j}(\mathcal{A}) = \frac{1}{2\pi\sigma} \left( \sum_{x \in \binom{[n]}{k}: j \in x, a \notin x} e^{-\frac{((\beta, x) - \delta)^2}{2\sigma^2}} - \sum_{x \in \binom{[n]}{k}: a \in x, j \notin x} e^{-\frac{((\beta, x) - \delta)^2}{2\sigma^2}} \right) = 0, \quad (12)$$

Next we show that these equalities can be valid together only on step functions  $\beta_j = 1/a$  for  $j \in [a]$ .

We tend  $\sigma \rightarrow 0$  and find, that to satisfy right hand side equality in (12) it is necessary to assume that the exponents from the left sum are equal to the corresponding exponents from the right sum i.e. for each given  $j \in [a - 1]$

$$((\beta, x) - \delta)^2 = ((\beta, y) - \delta)^2 \quad (13)$$

where  $x \in \binom{[n]}{k}, j \in x, y \in \binom{[n]}{k}, a \in y$  and  $x \setminus j, y \setminus a$  run over all sets of cardinality  $k - 1$  from  $[n] \setminus \{a, j\}$ . This is true, because values of  $\sigma$  can be chosen small.

We rewrite equalities (13) as follows:

$$\begin{aligned} & \beta_j^2 + (\beta_{j_1} + \dots + \beta_{j_{k-1}})^2 - 2\delta\beta_j - 2\delta(\beta_{j_1} + \dots + \beta_{j_{k-1}}) \\ & + 2\beta_j(\beta_{j_1} + \dots + \beta_{j_{k-1}}) = \\ & \beta_a^2 + (\beta_{m_1} + \dots + \beta_{m_{k-1}})^2 - 2\delta\beta_a - 2\delta(\beta_{m_1} + \dots + \beta_{m_{k-1}}) \\ & + 2\beta_a(\beta_{m_1} + \dots + \beta_{m_{k-1}}). \end{aligned}$$

Both sides of these equality over all permissible choices of  $j_1, \dots, j_{k-1}$  and  $m_1, \dots, m_{k-1}$  leads to the equality

$$\begin{aligned} & \binom{n-2}{k-1} (\beta_j^2 - 2\delta\beta_j) - 2\delta R + 2\beta_j R \\ & = \binom{n-2}{k-1} (\beta_a^2 - 2\delta\beta_a) - 2\delta R + 2\beta_a R \end{aligned} \quad (14)$$

where

$$R = \sum_{x \in \binom{[n]}{k-1} \setminus \{j, a\}} (\beta, x) = \binom{n-3}{k-2} \sum_{m \neq j, a} \beta_m = \binom{n-3}{k-2} (1 - \beta_j - \beta_a).$$

From (14) follows, that  $\beta_j$  can take at most two values:

$$\begin{aligned} \beta_j &= \beta_a, \\ \beta_j + \beta_a &= \gamma \triangleq 2 \frac{\delta - \frac{k-1}{n-2}}{1 - 2 \frac{k-1}{n-2}}. \end{aligned} \tag{15}$$

Next we show how we can eliminate the possibility that  $\beta_j$  takes second value. Assume at first that to each  $x$  such that  $|x \cap [a]| = p$  corresponds some  $y$  such that  $|y \cap [a]| = p$  for all  $x \in \binom{[n]}{k}$  and possible values of  $p$ . For given  $p$  we sum left and right sides of the relation (13) over  $x$  and corresponding  $y$  such that  $|x \cap [a]| = p$ . Then similar to the case of summation over all  $x$ , we obtain two possibilities:

$$\beta_j = \beta_a$$

or

$$\beta_j + \beta_a = 2 \frac{\delta - \frac{p-1}{a-2}}{1 - 2 \frac{p-1}{a-2}} \tag{16}$$

Because we can vary  $p$  it follows, that last equality for some  $p$  contradicts to the second equality from (15).

Now assume that for some  $b$

$$\beta_j = \begin{cases} \gamma - \beta_a, & j \leq b, \\ \beta_a, & j \in [b+1, a]. \end{cases} \tag{17}$$

Because  $\sum_j \beta_j = 1$  we have the following condition on  $\beta_a$  and  $\delta$ :

$$b\gamma + (a-2b)\beta_a = 1. \tag{18}$$

Let's  $\beta_j = \gamma - \beta_a$ . Assume also that for some  $x$  such that  $|x \cap [a]| = p$  corresponds some  $y$  such that  $|y \cap [a]| = q$  for some  $p \neq q$ . From (13) follows that there exists two possibilities

$$(\beta, x) = (\beta, y)$$



or

$$(\beta, x) + (\beta, y) = 2\delta. \quad (19)$$

Each of these equalities impose the condition- first equality the condition (for some integers  $p_1, p_2$ )

$$p_1\beta_a + p_2\gamma = 0$$

which can be inconsistent with equality (18) or together with equality (18) determine the value  $\delta_i$ .

From other side if equality (19) impose the condition (for some integers  $p_3, p_4$ )

$$p_3\beta_a + p_4\gamma = 2\delta. \quad (20)$$

It is possible that equality (18) together with equality (20) does not determine the value  $\delta$ . In this case we consider next three possibilities. First possibility that there exists  $x$  such that  $|x \cap [a]| = m$  (where  $m$  can be equal to  $p$  or  $q$ ) and there exists corresponding  $y$  such that  $|y \cap [a]| = v$  where  $v \neq p, q$

Second possibility is that to each  $x$  such that  $|x \cap [a]| = m$ , here  $m \neq p, q$  correspond to  $y$  such that  $|y \cap [a]| = m$ . In this, second case we return to the case which leads to the equalities (16) (because when  $a \geq 5$  the number of such  $m \neq p, q$  is greater than 1).

Third possibility is that some  $x$  such that  $|x \cap [a]|$ , here  $m \neq p, q$  corresponds to some  $y$  such that  $|y \cap [a]| \neq p, q, m$ .

If we have the first or third possibility, then we have one additional equation

$$q_3\beta_a + q_4\gamma = 2\delta \quad (21)$$

which together with (18) and (20) are inconsistent or determine unique value of  $\delta_i$ .

We see, that if  $b > 1$ , and  $\beta_j = \gamma - \beta_a > \beta_a$  when  $j \leq b$ , then value of  $\beta_i$  can take values only from some discrete finite set. Making small variation of  $\delta$  we can achieve the situation that neither of values of these functions are equal with true value of  $\delta$ . Once more we mention that such varying we can do always without violation the relation (11). Lemma is proved.

To prove lemma 1 we formulate the basic Optimization problem 1.

### Optimization problem 1.

Find maximum over the choice of  $\{\beta_j\}$  and  $\{\delta\}$  of the function

$$Y(\mathcal{A})$$

under the conditions

$$\sum_{m \in [\ell]} Z(x_m, \sigma) < \ell N - 1 + \mu_1 \quad (22)$$

and

$$\beta_{j+1} - \beta_j \leq 0, \quad (23)$$

where  $\{x_1, \dots, x_\ell\}$  runs over all sets of  $\ell$  different nonintersecting  $n$ -tuples from  $\binom{[n]}{k}$  and  $\mu_1$  is some small positive number less than 1.

Next we concentrate on the case only one  $\beta$ , the case which we actually need.

As we show before, parameters  $\beta, \delta$ , which maximize the value of  $Y(\mathcal{A})$ , maximize the value  $|\mathcal{A}|$  also and conditions (22) make it impossible the event that  $\mathcal{A}$  has  $\ell$ -matching.

We will show that conditional maximum of  $Y(\mathcal{A})$  achieves on the set of  $\beta$ , such that

$$\beta_j = \frac{1}{a}, \quad j \in [a]$$

for some  $a \in [k\ell - 1]$  and  $\beta_j = 0$  when  $j > a$ .

Next we make the following: we skip all conditions (22) except one. This only increase  $Y(\mathcal{A})$ . We choose only one set  $\{x\} = \{x_1, \dots, x_\ell\}$  of nonintersecting elements from  $\binom{[n]}{k}$ :

$$x_j = \{j, j + \ell, \dots, j + (k - 1)\ell\}, \quad j \in [\ell].$$

This set is forbidden to be included in  $\mathcal{A}$  by the inequality:

$$(\beta, x_j) \leq \delta, \quad (24)$$

Let's  $a = (m - 1)\ell + p$ , for some  $m \in [k]$  and  $p < \ell$  and  $\beta_j = 1/a$  when  $j \in [a]$ . The restriction (24) means that it is necessary and sufficient to choose  $\delta < \psi$  where

$$\psi = \frac{m - 1}{a} = \frac{m - 1}{(m - 1)\ell + p}. \quad (25)$$

For this choice  $\beta$  and  $\delta$ , which is sufficiently close to  $\psi$  we have

$$\sum_{j \geq m} \binom{(m - 1)\ell + p}{j} \binom{n - (m - 1)\ell - p}{k - j} \quad (26)$$

choices of admissible  $x \in \binom{[n]}{k}$ .

It is left to find the maximum over choices of values of  $a \in [k\ell - 1]$  of the sum from (26):

$$\begin{aligned} & \max_{a \in [k\ell - 1]} \sum_{j > a/\ell} \binom{a}{j} \binom{n-a}{k-j} \\ &= \max_{i \in [k]} \sum_{j \geq i} \binom{\ell i - 1}{j} \binom{n - \ell i + 1}{k-j}. \end{aligned} \tag{27}$$

Equality in (27) follows from the fact, that

$$\sum_{j > a/\ell} \binom{a}{j} \binom{n-a}{k-j}$$

decreases as  $a$  decreases from  $\ell i - 1$  to  $\ell(i - 1)$ .

It is left to proceed with Optimization Problem 1. Denote corresponding  $\sum_{m \in [\ell]} Z(x_m, \sigma)$  by  $Z(\{x\}, \sigma)$ .

Next we use Kuhn- Tucker necessary condition on  $\beta = \{\beta_j, j \in [a]\}$  on which achieved conditioned maximum of  $Y(\mathcal{A})$ .

Assume that  $\sum_{j=1}^a \beta_j = 1$  for some  $a \in [k\ell - 1]$ . It follows that  $\beta$ , on which achieved conditional maximum of  $Y(\mathcal{A})$  satisfies the equalities

$$Y'_{\beta_j}(\mathcal{A}) = \lambda Z'_{\beta_j}(\{x\}, \sigma) - 2\lambda_j, \tag{28}$$

$$Y'_\delta(\mathcal{A}) = \lambda Z'_\delta(\{x\}, \sigma), \tag{29}$$

where  $j \in [a - 1]$  and  $\lambda_j \geq 0$  satisfy relations  $\lambda_j(\beta_a - \beta_j) = 0$ . Here we relax conditions (23) to

$$\beta_a - \beta_j \leq 0.$$

Because  $Z'_\delta(\{x\}, \sigma) < 0$  these conditions allow to find all  $\beta_i$  on which achieved conditional maximum of  $Y(\mathcal{A})$ .

From (28), (29) follows the equations

$$Y'_{\beta_j}(\mathcal{A})Z'_\delta(\{x\}, \sigma) = Y'_\delta(\mathcal{A})Z'_{\beta_j}(\{x\}, \sigma) - 2\lambda_j Z'_\delta(\{x\}, \sigma). \tag{30}$$

Note that parameters  $\lambda_j$  arise when one takes into account conditions  $\beta_{i,a_i} - \beta_{i,j} \leq 0$  when considered Kuhn- Tucker conditions.

Next we start our analysis from  $j = 1$ . Here can be two cases  $\beta_1 > \beta_a$  or  $\beta_1 = \beta_a$ . In the second case we are done, we only need to choose  $\lambda_1 > 0$ .

As we will see later, we need to choose  $\lambda_1 = 0$  to keep the equality  $\beta_1 = \beta_a$ . Hence it is left to consider first case. Next we will show that we can eliminate it. In first case we have  $\lambda_1 = 0$  and thus equation (30) reduced to the equality

$$Y'_{\beta_1}(\mathcal{A})Z'_\delta(\{x\}, \sigma) = Y'_\delta(\mathcal{A})Z'_{\beta_1}(\{x\}, \sigma). \quad (31)$$

It follows that if

$$Z'_{\beta_1}(\{x\}, \sigma) = 0$$

then

$$Y'_{\beta_1}(\mathcal{A}) = 0.$$

Later we step by step collect the cases (for different  $j$ ) such that

$$Y'_{\beta_j}(\{x\}, \sigma) = 0$$

and make above analysis, having the cases (15), of the extremum of function  $Y(\mathcal{A})$  shows that in this case proper choice of  $\delta$  deliver the equality  $\beta_1 = \beta_a$  which contradicts to the proposition that  $\beta_1 > \beta_a$ . Thus we can assume that  $Z'_{\beta_1}(\{x\}, \sigma) \neq 0$ . Next we show that by proper choice of  $\delta$  we can make impossible for the equation (31) to be valid. We will do this similar to the method we apply to eliminate second value of  $\beta_j$  when determined unconditional extremum of  $Y(\mathcal{A})$ . We will show that value  $\beta_1$  which satisfies the equation (31) can takes at most two values and both of them can be eliminate by small shifting of  $\delta$ . From this will follow that we can exclude the possibility  $\beta_1 > \beta_a$  when determine the conditional maximum of  $Y(\mathcal{A})$  and can assume that  $\beta_1 = \beta_a$ . Next we make calculations which support these our considerations.

Remind the definition

$$\gamma = 2 \frac{\delta - \frac{k-1}{n-2}}{1 - 2 \frac{k-1}{n-2}}.$$

We sum the exponents of the terms in the both sides of (31) with proper signs similar as we do this when analyze unconditional maximum of  $Y(\mathcal{A})$  we obtain the relation

$$\begin{aligned} & \ell \left( \sum_{x \in \binom{[n]}{k}: 1 \in x, a \notin x} ((\beta, x) - \delta)^2 - \sum_{y \in \binom{[n]}{k}: a \in y, 1 \notin x} ((\beta, y) - \delta)^2 \right) \\ &= \binom{n}{k} \left( ((\beta, x) - \delta)^2 - ((\beta, y) - \delta)^2 \right), \end{aligned}$$

here in the right hand side  $x \in \{x\}$ ,  $1 \in x$  and  $y \in \{x\}$ ,  $a \in y$ . From here we obtain the quadratic (for  $\beta_1$ ) equality

$$\begin{aligned} & \ell \cdot \binom{n-2}{k-1} (\beta_1 - \beta_a) \left(1 - 2\frac{k-1}{n-2}\right) (\beta_1 + \beta_a - \gamma) \\ &= \binom{n}{k} ((\xi + \beta_1 - \delta)^2 - (\psi + \beta_a - \delta)^2). \end{aligned} \quad (32)$$

Here

$$\begin{aligned} \xi &= (\beta, x) - \beta_1, \\ \psi &= (\beta, y) - \beta_a \end{aligned}$$

for some  $x, y \in \{x\}$  such that  $1 \in x$  and  $a \in y$ .

Last relation generate solutions for  $\beta_1$  which depend on  $\delta$  is essentially algebraic or (possibly) linear function. Next we make the important remark. The number of positive or negative terms in the left hand side of (30) is  $\ell \binom{n-2}{k-1}$  and in the right hand side  $\binom{n}{k}$ . We can assume that  $k\ell < n$ , otherwise the answer of the matching problem is clear. Then to satisfy equality (30) we should assume that some two terms with different signs in the right hand side of (30) are equal. This gives the equation

$$((\beta, x) - \delta)^2 + ((\beta, x') - \delta)^2 = ((\beta, y) - \delta)^2 + ((\beta, y') - \delta)^2.$$

where  $x, y$  are chosen as in (33) (actually they are represented in the definitions of  $\xi$  and  $\psi$ ) and  $x'_1 = 1$ . From this equation it follows that  $\beta_1$  is essentially algebraic or linear function of  $\delta$ . It can be easily shown (it leads to some cumbersome calculations), that this function is differ from the function, generated by the equality (32). In both cases we can shift  $\delta$  in such a way that these two equations become inconsistent.

This proves that only the equation  $\beta_1 = \beta_a$  is possible. Step by step using similar considerations one can show that the only possible case is when  $\beta_1 = \dots = \beta_a$ . We only need to choose  $\lambda_j \geq 0$ . The choice is as follows.

There are (at most) three parts of  $[a]$ . First, where  $Z'_{\beta_j} > 0$ , second, where  $Z'_{\beta_j} = 0$  and third, where  $Z'_{\beta_j} < 0$ . In the first part we choose  $\lambda_j$ , such, that r.h.s. of the equation (28) is equal to zero, the same in the second part. In the third part we choose  $\lambda_j$  such, that r.h.s of (28) is equal  $\pm$  l.h.s. of this equation, hence we obtain in this case identity or that  $Y'_{\beta_j}(\mathcal{A}) = 0$ , which is consistent with the solution  $\beta_j = \beta_a$ .

The last we should show is that we can choose  $\delta$  in such a way that all  $\lambda_j \geq 0$ . But it is obvious from the choice considered above- it is necessary to choose  $\delta$  sufficiently close to  $(\beta, x)$  when  $(\beta, x) \neq (\beta, y)$  and note that  $(\beta, x)$ , when the last inequality is valid and  $\beta_j = \frac{1}{a}$  when  $j \in [a]$ , does not depend on the choice of  $x \in \{x\}$ .

Lemma 1 is proved.

### Section III. Proof of the Conjecture 2

To prove conjecture 2 we will follow the same procedure as in the previous section. First we formulate the optimization problem:

#### Optimization Problem 2.

Maximize over the choice of  $\{\beta_{i,j}\}$  and  $\{\delta_i\}$  the function

$$Y(\mathcal{A})$$

under the restrictions

$$\sum_{m \in [s]} Z(x_m, \sigma) < Ns - 1 + \mu_2,$$

$$\beta_{i,j+1} \leq \beta_{i,j},$$

where  $(x_1, \dots, x_s)$  runs over all subsets of  $s$  different  $n$ -tuples from  $\binom{[n]}{k}$  which are not  $s$ -wise  $t$ -intersecting and  $\mu_2 > 0$  is small number.

This problem has some difference from the previous problem, because we should consider the case when  $\beta_{i,j}$  can be positive for all  $j \in [n-1]$ , not only  $j \leq ks-1$  as in the previous case. But literally the same procedure with one value  $Z(\{x_m\}, \sigma)$  for the set of  $x_m$ , defined below, shows that for each  $i \in [N]$  all positive  $\beta_{i,j}$  should be equal. We skip the details.

Let  $a$  be the number of positive (equal)  $\beta_j = 1/a$ . As in the previous section we leave only one restriction from the optimization problem 2. Restriction which is generated by the following  $s$  elements:

$$\begin{aligned} x_1 &= (1, 2, \dots, t-1, t+1, \dots, t+s-1, t+s+1, \dots, t+2s-1, t+2s+1, \\ &\quad \dots, t+3s-1, \dots, c(1)), \\ x_m &= (1, 2, \dots, t, c_1(m), c_2(m), \dots, c_d(m), c(m)), \quad m = 2, \dots, s, \end{aligned}$$

where  $c_j(m)$  is cyclic shifting of  $s$ -tuple  $(t + (j-1)s + 1, \dots, t + js - 1, \emptyset)$  on  $m-1$  positions to the right (shifting here actually means that we put  $\emptyset$  consequently on each position starting from the leftmost position and renumber the elements of the sequence giving them the number of the right neuborhood, at the beginning empty set has number  $t + js$ ), except, possibly, last  $c(m)$  which is, possibly, reduced (with fewer number of elements), because we have restriction on the number  $k$  of elements in  $x_m$ . We choose

$c(1) = (t + ds + 1, \dots, k, \emptyset, \dots, \emptyset)$ ,  $c(2) = (t + ds + 1, \dots, k + 1, \emptyset, \dots, \emptyset)$  where the length of tuples  $c(1), c(2)$  is  $s + 1$  and  $c(m)$ ,  $m = \{3, \dots, s\}$  is obtained from  $c(1)$  by the same shifting as  $c_j(m)$  from  $c_j(1)$ . Set  $\{x_m; m \in [s]\}$  is not  $s$ -wise  $t$ -intersecting set, because in  $n$ -tuple  $x_1$  does not have element  $t$  and apart from the elements  $[t]$  these  $n$ -tuples does have even one element in common.

Now as before assume that restriction

$$(\beta, x) > \delta$$

forbids this set of  $n$ -tuples. It is easy to see, that it is impossible that  $a \in [t - 1]$ , because in this case should be  $\delta \geq 1$  and  $\mathcal{A} = \emptyset$ . If  $a = t$ , then we should choose  $\delta \in [(t - 1)/t, 1)$ . This choice forbids  $n$ -tuple  $x_1$  as a member of  $\mathcal{A}$  and allow other  $x_i$ . If  $a \in \{t + ps + 1, \dots, t + (p + 1)s\}$ , then we should choose  $\delta \in [(a - p - 1)/a, (a - p)/a)$ . This choice of  $\delta_i$  forbids at least one  $n$ -tuple  $x_i$ , but does not make any further restrictions. At last if  $(s - 1)$  does not divide  $(k - t)$ , then for  $a \in \{t + ds + 1, \dots, k + d\}$  we should choose  $\delta \in [(a - d - 1)/a, (a - d)/a)$  and if  $a > k + d$ , then even allowing all  $n$ -tuples  $x$  such that  $|x \cap [a]| = k$  does not warranty  $s$ -wise  $t$ -intersection, hence  $a \leq k + d$ .

Collecting these possibilities for the choice of of pairs  $(a, \delta_i)$  together we see that  $(p < s)$

$$N(s, n, k, t) \leq \max_{a=t+sp+r} \sum_{i \geq t+(s-1)p+r} \binom{a}{i} \binom{n-a}{k-i}. \quad (33)$$

It is enough to make the optimization in (33) only over  $a$  such that  $s|(a - t)$ . Indeed it easily follows from the inequality

$$\sum_{j \geq i-1} \binom{a-1}{j} \binom{n-a+1}{k-j} \geq \sum_{j \geq i} \binom{a}{j} \binom{n-a}{k-j}$$

which can be proved by using the identity

$$\binom{k}{m} = \binom{k-1}{m} + \binom{k-1}{m-1}.$$



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